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Ground state properties of one-dimensional Bose–Fermi mixtures

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Abstract

Bose–Fermi mixtures in one dimension are studied in detail on the basis of an exact solution. Corresponding to three possible choices of the reference state in the quantum inverse scattering method, three sets of Bethe-ansatz equations are derived explicitly. The features of the ground state and low-lying excitations are investigated. The ground state phase diagram caused by the external field and chemical potential is obtained.

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1. Introduction

The study of exactly solvable models has been an important topic for four decades because the perturbative approach is inapplicable for strongly correlated systems. 'Particles with δ-function interaction in one dimension' is a simple but interesting model. Since Lieb and Liniger [1] first solved a spinless Bose system with δ-function interaction, there has been much more progress [2–12] in this field. In particular, when solving the 2-component Fermi system, Yang [6] proposed the well-known Yang–Baxter equation which had a significant impact on both physics and mathematics. As the 2-component system is mostly associated with 'spin-1/2' particles that are conventionally referred to as the Fermi system, the coordinate Bethe-ansatz has not been applied to the 2-component Bose system till recently [13], which is motivated by a spinor Bose–Einstein condensate in magnetically trapped ^{87}Rb [14].

Recent observation of the superfluid to Mott insulator transition in ultracold atoms in an optical lattice [15] stimulated research interests related to strongly correlated atomic gases. Most recently, the physics of ultracold Bose–Fermi mixtures [16] such as ^{7}Li–^{6}Li or ^{87}Rb–^{40}K mixtures has become a remarkable topic [17–21]. It is therefore worthwhile to investigate the features of the Bose–Fermi system on the basis of exact solutions. The mixed system of bosons and fermions with δ-function interaction was discussed earlier in [10] where the ground state energy and gapless fermionic excitations are calculated in the thermodynamic limit. However, as we are aware, the ground state phase diagram under the influence of external fields and

chemical potentials has not yet been studied. These properties have become more and more important nowadays due to the rapid progress in the field of cold atomic physics.

In this paper, we study a one-dimensional cold atomic system of Bose–Fermi mixtures systematically. Our paper is organized as follows. In the following section, we introduce the model and derive its secular equation. In section 3, we diagonalize the secular equation by means of the quantum inverse scattering method (QISM) for three cases. Consequently, three different kinds of nested Bethe-ansatz equations are obtained. In section 4, we explicitly analyse the ground state and the possible low-lying excitations. The energy–momentum spectrum for each excitation is calculated numerically from the Bethe-ansatz equation. In the last section, we study the system under the influence of magnetic fields and chemical potentials for the particles to obtain the phase diagram.

2. The model and its secular equation

We consider a mixture of cold Bose gas and Fermi gas in one dimension. The Hamiltonian of the system is described by the Gross–Pitaevskii functional

$$\mathbb{H} = \int \left(\sum_a \partial_x \Psi_a^* \partial_x \Psi_a + c \sum_{a,b} \Psi_a^* \Psi_a \Psi_b^* \Psi_b \right) \mathrm{d}x \tag{1}$$

where the natural unit is adopted for simplicity, c denotes the interaction strength and $a, b = 1, 2, 3$ refer to the three components of $SU(1|2)$ fields. This is an isotropic case of the model considered by Cazallia *et al* [22] where an approximation method was employed. We just consider this case because the anisotropic case is unintegrable. Among these three fields, two obey the anti-commutation relation and one obeys the commutation relation. It is convenient to consider the states that span a Hilbert space of N particles

$$|\psi\rangle = \sum_{x_1, x_2, \ldots} \psi_{a_1, \ldots, a_N}(x_1, x_2, \ldots, x_N) \Psi_{a_1}(x_1) \cdots \Psi_{a_N}(x_N)|0\rangle.$$

The eigenvalue problem $\mathbb{H}|\psi\rangle = E|\psi\rangle$ becomes an N-particle quantum mechanical problem with the first quantized Hamiltonian,

$$\mathcal{H} = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2c \sum_{i<j} \delta(x_i - x_j). \tag{2}$$

Such a system can be solved by means of the Bethe-ansatz approach. Here we give a brief description of the main idea of this approach. In the domain $x_i \neq x_j$, the Hamiltonian (2) reduces to that for free particles and its eigenfunctions are therefore just superpositions of plane waves. When two particles collide, a scattering process occurs, which is supposed to be a pure elastic process, i.e., exchange of their momenta. So, for a given momentum $k = (k_1, k_2, \ldots, k_N)$ the scattering momenta include all permutations of the components of k. Because the Hamiltonian is invariant under the action of the permutation group S_N, one can adopt the following Bethe-ansatz wavefunction,

$$\psi_a(x) = \sum_{P \in S_N} A_a(P, Q)\, \mathrm{e}^{\mathrm{i}(Pk|Qx)} \tag{3}$$

where $a = (a_1, a_2, \ldots, a_N)$, with a_j denoting the $SU(1|2)$ component of the jth particles; Pk denotes the image of a given $k := (k_1, k_2, \ldots, k_N)$ by a mapping $P \in S_N$; $(Pk|Qx) = \sum_{j=1}^{N} (Pk)_j (Qx)_j$; and the coefficients $A(P, Q)$ are functions of P and Q where Q denotes the permutation such that $0 < x_{Q_1} < x_{Q_2} < \cdots < x_{Q_N} < L$. For the Bose–Fermi mixture, the

wavefunction should be either symmetric or antisymmetric under permutation Π^j depending on whether they involve Bose labels or Fermi labels:

$$(\Pi^j \psi)_a(x) = \pm \psi_{\Pi^j a}(x). \tag{4}$$

The δ-function term in the Hamiltonian contributes a boundary condition across the hyperplane $x_{Q_j} = x_{Q_{j+1}}$. Substituting the Bethe-ansatz wavefunction into this boundary condition and using the continuity condition together with the permutation symmetry, we obtain the following relation,

$$A_a(\Pi^j P, Q) = \frac{\mathrm{i}[(Pk)_j - (Pk)_{j+1}]P^j + c}{\mathrm{i}[(Pk)_j - (Pk)_{j+1}]P^j - c} A_a(P, Q) \tag{5}$$

where P^j are the permutations between particles at x_{Q_j} and $x_{Q_{j+1}}$, which is given in the appendix for concrete choice of Bose–Fermi labels. For example, if we consider the wavefunctions of two particles, because of the different exchange symmetries, the wavefunction of two bosons is $\frac{1}{\sqrt{2}}(|\psi_1\rangle|\psi_2\rangle + |\psi_2\rangle|\psi_1\rangle)$, and $\frac{1}{\sqrt{2}}(|\psi_1\rangle|\psi_2\rangle - |\psi_2\rangle|\psi_1\rangle)$ for two fermions. The permutations for two bosons are

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and $-P$ for two fermions. The matrix relating to the various amplitudes in the same region given in equation (5) is conventionally called the S-matrix

$$S^{j,j+1} = \frac{\mathrm{i}[(Pk)_j - (Pk)_{j+1}]P^j + c}{\mathrm{i}[(Pk)_j - (Pk)_{j+1}]P^j - c}.$$

The amplitudes in region Q and in its adjacent region Q' are related by the R-matrix $R = PS$,

$$A_{a_1 \cdots a_i \cdots a_j \cdots a_N}(Q') = (R^{ij})^{b_1 \ldots b_N}_{a_1 \ldots a_N} A_{b_1 \ldots b_N}(Q) = (R^{ij})^{b_i b_j}_{a_i a_j} A_{a_1 \cdots b_i \cdots b_j \cdots a_N}(Q).$$

If x is a point in the region $C(Q^{(i)})$, then $x' = (x_1, \ldots, x_{Q_1} + L, \ldots, x_N)$ is a point in the region $C(\gamma Q^{(i-1)})$ with $\gamma = \Pi^{N-1}\Pi^{N-2} \cdots \Pi^2 \Pi^1$. Thus the periodic boundary condition imposes a relation between the wavefunctions defined on $C(Q^{(i)})$ and $C(\gamma Q^{(i-1)})$. Writing out this relation in terms of equation (3), we find that the periodic boundary conditions are guaranteed provided that $A(P; \gamma Q^{(i-1)})\, \mathrm{e}^{\mathrm{i}(Pk)_1 L} = A(P; Q^{(i)})$. After applying the R-matrix successively, we obtain the following secular equation:

$$R^{Q1,QN} \cdots R^{Q1,Q(i+1)} R^{Q1,Qi} \cdots R^{Q1,Q2} A(P; Q^{(i)}) = \mathrm{e}^{-\mathrm{i}(Pk)_1 L} A(P; Q^{(i)}). \tag{6}$$

3. Diagonalization by quantum inverse scattering method

To determine the spectrum, we should diagonalize the secular equation (6). This can be done by diagonalizing the operator product on the left-hand side of equation (6), namely, solving the eigenvalues of the operator

$$(T_j)^{b_1 b_2 \cdots b_N}_{a_1 a_2 \cdots a_N} = (R^{jj-1} \cdots R^{j1} R^{jN} \cdots R^{jj+1})^{b_1 b_2 \cdots b_N}_{a_1 a_2 \cdots a_N} \tag{7}$$

where

$$R^{ij} = \frac{(\alpha_i - \alpha_j) I^{ij} - \mathrm{i}c P^{ij}}{\alpha_i - \alpha_j + \mathrm{i}c}. \tag{8}$$

Since they satisfy the Yang–Baxter relation

$$R^{kj}(\alpha-\beta)R^{ki}(\alpha)R^{ji}(\beta)=R^{ji}(\beta)R^{ki}(\alpha)R^{kj}(\alpha-\beta) \tag{9}$$

the diagonalization can be carried out by means of QISM. For equation (8), a 9×9 monodromy can be defined in the conventional way,

$$(T)^{b_1b_2\cdots b_N,\upsilon}_{a_1a_2\cdots a_N,\mu}=\sum_{S_1S_2\cdots S_{N-1}}(R^{1A})^{b_1,S_1}_{a_1,\mu}(R^{2A})^{b_2,S_2}_{a_2,S_1}\cdots(R^{NA})^{b_N,\upsilon}_{a_N,S_{N-1}}, \tag{10}$$

which can be written as a 3×3 matrix in the auxiliary space,

$$T=\begin{pmatrix}A & B_1 & B_2\\ C_1 & D_{11} & D_{12}\\ C_2 & D_{21} & D_{22}\end{pmatrix}, \tag{11}$$

in which every matrix element is an operator in quantum space. It obeys the following RTT relations,

$$R(\lambda-\mu)T_1(\lambda)T_2(\mu)=T_2(\mu)T_1(\lambda)R(\lambda-\mu) \tag{12}$$

where $T_1=T_1\otimes I$, $T_2=I\otimes T_2$ with I the 3×3 unitary matrix in quantum space.

Since the Bose–Fermi mixture is an $SU(1|2)$ supersymmetric system, the application of QISM becomes complicated. In the $SU(3)$ case a unique nested Bethe-ansatz equation was derived [23]. In the present case, however, there are three possibilities of choosing the reference state ('pseudo-vacuum') and the successive orders of the other states, and hence three types of nested Bethe-ansatz equations have to be derived. In the following, we will consider those three cases.

3.1. BFF case

We first choose the Bose state as the reference state $|1\rangle$, and the other two states $|2\rangle$ and $|3\rangle$ are Fermi states, then the permutation operator is easily written out (see P_1 in (A.1)). This case was once noticed by Sutherland [9] in the lattice model. Consequently, the RTT relation (12) gives rise to two commutation relations between A and B, and eight commutation relations between B and D. We can write them in the form of a tensor product,

$$A(\lambda)\otimes(B_1(\mu)B_2(\mu))=\frac{k(\mu-\lambda)}{b(\mu-\lambda)}(B_1(\mu)B_2(\mu))\otimes A(\lambda)-\frac{d(\mu-\lambda)}{b(\mu-\lambda)}(B_1(\lambda)B_2(\lambda))\otimes A(\mu) \tag{13}$$

$$\begin{aligned}\begin{pmatrix}D_{11}(\lambda) & D_{12}(\lambda)\\ D_{21}(\lambda) & D_{22}(\lambda)\end{pmatrix}\otimes(B_1(\mu)B_2(\mu))&=(B_1(\mu)B_2(\mu))\otimes\begin{pmatrix}D_{11}(\lambda) & D_{12}(\lambda)\\ D_{21}(\lambda) & D_{22}(\lambda)\end{pmatrix}\\ &\times\frac{a(\lambda-\mu)}{b(\lambda-\mu)}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -\frac{d}{a} & \frac{b}{a} & 0\\ 0 & \frac{b}{a} & -\frac{d}{a} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\\ &-\frac{d(\lambda-\mu)}{b(\lambda-\mu)}(B_1(\lambda)B_2(\lambda))\otimes\begin{pmatrix}D_{11}(\mu) & D_{12}(\mu)\\ D_{21}(\mu) & D_{22}(\mu)\end{pmatrix}\end{aligned} \tag{14}$$

where we used the definition

$$k(\alpha)=\frac{\alpha-\mathrm{i}c}{\alpha+\mathrm{i}c},\qquad a(\alpha)=1,\qquad b(\alpha)=\frac{\alpha}{\alpha+\mathrm{i}c},\qquad d(\alpha)=\frac{-\mathrm{i}c}{\alpha+\mathrm{i}c}.$$

It is convenient to write the R-matrix in auxiliary space in terms of the generators of the corresponding Lie algebra (see the appendix),

$$R(\alpha) = \begin{pmatrix} \frac{\alpha}{\alpha+\mathrm{i}c} I - \mathrm{i}c \frac{2H_1+H_2+I}{3} & \frac{-\mathrm{i}c}{\alpha+\mathrm{i}c} E_{-\alpha_1} & \frac{-\mathrm{i}c}{\alpha+\mathrm{i}c} E_{-(\alpha_1+\alpha_2)} \\ \frac{-\mathrm{i}c}{\alpha+\mathrm{i}c} E_{\alpha_1} & \frac{\alpha}{\alpha+\mathrm{i}c} I + \frac{\mathrm{i}c}{\alpha+\mathrm{i}c} \frac{H_2-H_1+I}{3} & \frac{\mathrm{i}c}{\alpha+\mathrm{i}c} E_{-\alpha_2} \\ \frac{-\mathrm{i}c}{\alpha+\mathrm{i}c} E_{(\alpha_1+\alpha_2)} & \frac{\mathrm{i}c}{\alpha+\mathrm{i}c} E_{\alpha_2} & \frac{\alpha}{\alpha+\mathrm{i}c} I + \frac{\mathrm{i}c}{\alpha+\mathrm{i}c} \frac{I-H_1-2H_2}{3} \end{pmatrix}, \tag{15}$$

and to define a pseudo-vacuum as N particles in the highest weight state of the $SU(1|2)$ systems, i.e.,

$$|\phi\rangle = \prod_{i=1}^{N} \otimes |1\rangle \tag{16}$$

with $|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $|2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $|3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. We have $H_1|1\rangle = |1\rangle$, $E_{-\alpha_1}|1\rangle = |2\rangle$, $E_{-(\alpha_1+\alpha_2)}|1\rangle = |3\rangle$, and null if the other operators act on the state $|1\rangle$. Thus the vertex (15) becomes triangular when acting on the highest weight state. The eigenvalues of diagonal terms of the monodromy can be obtained as the products of the values in each quantum space, and hence the eigenvalues of operators $A(k)$ and $D(k)$ are

$$A(k)|\phi\rangle = \prod_{l=1}^{N} \frac{k - k_l - \mathrm{i}c}{k - k_l + \mathrm{i}c} |\phi\rangle \tag{17}$$

$$D(k)|\phi\rangle = \prod_{l=1}^{N} \frac{k - k_l}{k - k_l + \mathrm{i}c} |\phi\rangle. \tag{18}$$

Here

$$D(k) = \begin{pmatrix} D_{11}(k) & D_{12}(k) \\ D_{21}(k) & D_{22}(k) \end{pmatrix} \tag{19}$$

is the monodromy of $SU(2)$ algebra nested in $SU(1|2)$.

Employing the lowering operator $B(\lambda)$ acting on the pseudo-vacuum, we construct an eigenstate

$$|\omega\rangle = B(\lambda_1)B(\lambda_2)\cdots B(\lambda_M)|\phi\rangle \tag{20}$$

where $B(\lambda)$ refers to one of the $B_1(\lambda)$ or $B_2(\lambda)$ in the monodromy (11). This state can be used to diagonalize the secular equation. The eigenvalue on the left-hand side of the secular equation (6) can be written as the eigenvalue of the trace of the $SU(1|2)$ monodromy matrix:

$$\begin{aligned} &\Lambda_{SU(1|2)}(k; \lambda_1, \lambda_2, \ldots, \lambda_M)|\omega\rangle \\ &\quad = \mathrm{Tr}(T)|\omega\rangle = (A(k) + \mathrm{Tr}(D(k)))|\omega\rangle \\ &\quad = \prod_{l=1}^{N} \frac{k - k_l - \mathrm{i}c}{k - k_l + \mathrm{i}c} \prod_{\alpha=1}^{M} \frac{k(\lambda_\alpha - k)}{b(\lambda_\alpha - k)} |\omega\rangle \\ &\qquad + \prod_{l=1}^{N} \frac{k - k_l}{k - k_l + \mathrm{i}c} \prod_{\alpha=1}^{M} \frac{a(k - \lambda_\alpha)}{b(k - \lambda_\alpha)} \cdot \Lambda'_{SU(2)}|\omega\rangle. \end{aligned} \tag{21}$$

From this nested Bethe-ansatz structure we can see that there is an $SU(2)$ substructure in the $SU(1|2)$ system when the boson state is chosen as the reference state. In terms of the $\check{R}$-matrix of $SU(2)$ appearing in equation (14)

$$\check{r} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{d}{a} & \frac{b}{a} & 0 \\ 0 & \frac{b}{a} & -\frac{d}{a} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{22}$$

and the permutation matrix of $SU(2)$

$$p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{23}$$

we have the R-matrix $r = p \cdot \check{r}$. Writing it out in auxiliary space in terms of spin operators in quantum space,

$$r(\alpha) = \begin{pmatrix} \frac{\alpha+\mathrm{i}c/2}{\alpha+\mathrm{i}c} I + \frac{\mathrm{i}c/2}{\alpha+\mathrm{i}c}\sigma_z & \frac{\mathrm{i}c}{\alpha+\mathrm{i}c}\sigma^- \\ \frac{\mathrm{i}c}{\alpha+\mathrm{i}c}\sigma^+ & \frac{\alpha+\mathrm{i}c/2}{\alpha+\mathrm{i}c} I - \frac{\mathrm{i}c/2}{\alpha+\mathrm{i}c}\sigma_z \end{pmatrix},$$

we obtain the fundamental commutation relation from the RTT relations, $r \cdot \tilde{T}_1 \cdot \tilde{T}_2 = \tilde{T}_2 \cdot \tilde{T}_1 \cdot r$, as follows,

$$\begin{aligned} A'(\lambda)B'(\mu) &= \frac{a'(\mu-\lambda)}{b'(\mu-\lambda)} B'(\mu)A'(\lambda) - \frac{d'(\mu-\lambda)}{b'(\mu-\lambda)} B'(\lambda)A'(\mu) \\ D'(\lambda)B'(\mu) &= \frac{a'(\lambda-\mu)}{b'(\lambda-\mu)} B'(\mu)D'(\lambda) - \frac{d'(\lambda-\mu)}{b'(\lambda-\mu)} B'(\lambda)D'(\mu) \end{aligned} \tag{24}$$

where

$$a'(\alpha) = 1, \qquad b'(\alpha) = \frac{b}{a} = \frac{\alpha}{\alpha + \mathrm{i}c}, \qquad d'(\alpha) = -\frac{d}{a} = \frac{\mathrm{i}c}{\alpha + \mathrm{i}c}$$

and the $D(k)$ matrix, a sub-matrix of the $SU(1|2)$ matrix, is regarded as the $SU(2)$ monodromy, namely

$$\widetilde{T}(k) = \begin{pmatrix} A'(k) & B'(k) \\ C'(k) & D'(k) \end{pmatrix}$$

where $A'(k) = D_{11}(k)$, $B'(k) = D_{12}(k)$, $C'(k) = D_{21}(k)$, $D'(k) = D_{22}(k)$.

According to the procedure of QISM [12], the pseudo-vacuum is defined as the product of the highest weight states of $SU(2)$ $|\phi'\rangle = \prod^M \binom{1}{0}$ which fulfils

$$A'(k)|\phi'\rangle = \prod_{\alpha=1}^{M} a'(k-\lambda_\alpha)|\phi'\rangle, \qquad D'(k)|\phi'\rangle = \prod_{\alpha=1}^{M} b'(k-\lambda_\alpha)|\phi'\rangle.$$

In terms of the lowering operator $B'(k-\lambda_\alpha)$ in $SU(2)$ monodromy, one can construct a general state

$$|\omega'\rangle = B'(k-\lambda_1)B'(k-\lambda_2)\cdots B'(k-\lambda_{M'})|\phi'\rangle. \tag{25}$$

Using the fundamental commutation relations (24), one obtains that

$$\begin{aligned}\Lambda'_{SU(2)}(k;\mu_1,\mu_2,\ldots,\mu_{M'})|\omega'\rangle &= \mathrm{Tr}(\widetilde{T}(k))|\omega'\rangle \\ &= \left(\prod_{\alpha=1}^{M} a'(k-\lambda_\alpha)\prod_{\beta=1}^{M'}\frac{a'(\mu_\beta-k)}{b'(\mu_\beta-k)} + \prod_{\alpha=1}^{M} b'(k-\lambda_\alpha)\prod_{\beta=1}^{M'}\frac{a'(k-\mu_\beta)}{b'(k-\mu_\beta)}\right)|\omega'\rangle \\ &= \left(\prod_{l=\beta}^{M'}\frac{\mu_\beta-k+\mathrm{i}c}{\mu_\beta-k} + \prod_{\alpha=1}^{M}\frac{k-\lambda_\alpha}{k-\lambda_\alpha+\mathrm{i}c}\prod_{\beta=1}^{M'}\frac{k-\mu_\beta+\mathrm{i}c}{k-\mu_\beta}\right)|\omega'\rangle. \end{aligned} \tag{26}$$

The unwanted terms vanish as long as the following equations hold:

$$-\prod_{\beta=1}^{M'}\frac{\mu_c-\mu_\beta-\mathrm{i}c}{\mu_c-\mu_\beta+\mathrm{i}c} = \prod_{\alpha=1}^{M}\frac{\mu_c-\lambda_\alpha}{\mu_c-\lambda_\alpha+\mathrm{i}c}. \tag{27}$$

As a result, equation (21) becomes

$$\begin{aligned}&\Lambda_{SU(1|2)}(k;\lambda_1,\lambda_2,\ldots,\lambda_M) \\ &= \prod_{l=1}^{N}\frac{k-k_l-\mathrm{i}c}{k-k_l+\mathrm{i}c}\prod_{\alpha=1}^{M}\frac{k(\lambda_\alpha-k)}{b(\lambda_\alpha-k)} + \prod_{l=1}^{N}\frac{k-k_l}{k-k_l+\mathrm{i}c}\prod_{\alpha=1}^{M}\frac{a(k-\lambda_\alpha)}{b(k-\lambda_\alpha)}\cdot\Lambda'_{SU(2)} \\ &= \prod_{l=1}^{N}\frac{k-k_l-\mathrm{i}c}{k-k_l+\mathrm{i}c}\prod_{\alpha=1}^{M}\frac{\lambda_\alpha-k-\mathrm{i}c}{\lambda_\alpha-k} + \prod_{l=1}^{N}\frac{k-k_l}{k-k_l+\mathrm{i}c}\prod_{\alpha=1}^{M}\frac{k-\lambda_\alpha+\mathrm{i}c}{k-\lambda_\alpha} \\ &\quad\times\left(\prod_{\beta=1}^{M'}\frac{\mu_\beta-k+\mathrm{i}c}{\mu_\beta-k} + \prod_{\alpha=1}^{M}\frac{k-\lambda_\alpha}{k-\lambda_\alpha+\mathrm{i}c}\prod_{\beta=1}^{M'}\frac{k-\mu_\beta+\mathrm{i}c}{k-\mu_\beta}\right). \end{aligned} \tag{28}$$

To get rid of the unwanted terms in the expansion, the following equations need to be satisfied:

$$1 = -\prod_{l=1}^{N}\frac{\lambda_\gamma-k_l-\mathrm{i}c}{\lambda_\gamma-k_l}\prod_{\beta=1}^{M'}\frac{\mu_\beta-\lambda_\gamma}{\mu_\beta-\lambda_\gamma+\mathrm{i}c}. \tag{29}$$

It is convenient to redefine the parameter λ'_γ by $\lambda_\gamma - \mathrm{i}c/2$. Equations (29) and (27) for the complete cancellation of the unwanted terms appearing in both procedures, together with the relation resulting from periodic boundary conditions, $\mathrm{e}^{-\mathrm{i}k_\alpha L} = \Lambda_{SU(1|2)}(k;\lambda_1,\lambda_2,\ldots,\lambda_M)$ give rise to the Bethe-ansatz equations

$$\begin{aligned}\mathrm{e}^{\mathrm{i}k_j L} &= -\prod_{l=1}^{N}\frac{k_j-k_l+\mathrm{i}c}{k_j-k_l-\mathrm{i}c}\prod_{\alpha=1}^{M}\frac{k_j-\lambda_\alpha-\mathrm{i}c/2}{k_j-\lambda_\alpha+\mathrm{i}c/2} \\ 1 &= -\prod_{l=1}^{N}\frac{\lambda_\gamma-k_l-\mathrm{i}c/2}{\lambda_\gamma-k_l+\mathrm{i}c/2}\prod_{\beta=1}^{M'}\frac{\lambda_\gamma-\mu_\beta+\mathrm{i}c/2}{\lambda_\gamma-\mu_\beta-\mathrm{i}c/2} \\ 1 &= -\prod_{\alpha=1}^{M}\frac{\mu_c-\lambda_\alpha-\mathrm{i}c/2}{\mu_c-\lambda_\alpha+\mathrm{i}c/2}\prod_{\beta=1}^{M'}\frac{\mu_c-\mu_\beta+\mathrm{i}c}{\mu_c-\mu_\beta-\mathrm{i}c} \end{aligned} \tag{30}$$

which determine the spectrum of the $SU(1|2)$ system.

3.2. FBF case

We now consider the second case, in which the Bose state is chosen as the second state, while the first and the third are Fermi states. From the permutation operator P_2 (see (A.2)), we can

get the R-matrix (8). Using the same monodromy as in equation (11) and the RTT relation (12), we get the following communication relations:

$$A(\lambda)\otimes(B_1(\mu)B_2(\mu)) = \frac{a(\mu-\lambda)}{b(\mu-\lambda)}(B_1(\mu)B_2(\mu))\otimes A(\lambda) - \frac{d(\mu-\lambda)}{b(\mu-\lambda)}(B_1(\lambda)B_2(\lambda))\otimes A(\mu)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{31}$$

$$\begin{pmatrix}D_{11}(\lambda) & D_{12}(\lambda)\\ D_{21}(\lambda) & D_{22}(\lambda)\end{pmatrix}\otimes(B_1(\mu)B_2(\mu)) = (B_1(\mu)B_2(\mu))\otimes\begin{pmatrix}D_{11}(\lambda) & D_{12}(\lambda)\\ D_{21}(\lambda) & D_{22}(\lambda)\end{pmatrix}\frac{k(\lambda-\mu)}{b(\lambda-\mu)}\cdot\check{r} - \frac{d(\lambda-\mu)}{b(\lambda-\mu)}(B_1(\lambda)B_2(\lambda))\otimes\begin{pmatrix}D_{11}(\mu) & D_{12}(\mu)\\ D_{21}(\mu) & D_{22}(\mu)\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{32}$$

In this case, we can see that there is a nested $SU(1|1)$ substructure in the $SU(1|2)$ system. The $\check{R}$-matrix of $SU(1|1)$ in equation (32) reads

$$\check{r} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \frac{d}{k} & \frac{b}{k} & 0\\ 0 & \frac{b}{k} & \frac{d}{k} & 0\\ 0 & 0 & 0 & \frac{a}{k}\end{pmatrix}. \tag{33}$$

In this $SU(1|1)$ substructure, the Bose state is chosen as the highest weight state $|\phi'\rangle = \prod^{M}\binom{1}{0}$ when the QISM [12] is applied. We can obtain the Bethe-ansatz equation by the similar procedure applied in the previous case,

$$\begin{aligned} \mathrm{e}^{\mathrm{i}k_jL} &= -\prod_{\alpha=1}^{M}\frac{k_j-\lambda_\alpha+\mathrm{i}c/2}{k_j-\lambda_\alpha-\mathrm{i}c/2}\\ 1 &= -\prod_{l=1}^{N}\frac{\lambda_\gamma-k_l-\mathrm{i}c/2}{\lambda_\gamma-k_l+\mathrm{i}c/2}\prod_{\beta=1}^{M'}\frac{\mu_\beta-\lambda_\gamma-\mathrm{i}c/2}{\mu_\beta-\lambda_\gamma+\mathrm{i}c/2}\\ 1 &= -\prod_{\alpha=1}^{M}\frac{\mu_c-\lambda_\alpha+\mathrm{i}c/2}{\mu_c-\lambda_\alpha-\mathrm{i}c/2}. \end{aligned} \tag{34}$$

3.3. FFB case

We turn to the case where the Bose state is chosen as the third state, and one of the Fermi states as the reference state, the other Fermi state as the second state. In terms of the permutation matrix P_3 (see (A.3)), we can get the R-matrix (8). The RTT relation (12) gives rise to the following communication relations:

$$A(\lambda)\otimes(B_1(\mu)B_2(\mu)) = \frac{a(\mu-\lambda)}{b(\mu-\lambda)}(B_1(\mu)B_2(\mu))\otimes A(\lambda) - \frac{d(\mu-\lambda)}{b(\mu-\lambda)}(B_1(\lambda)B_2(\lambda))\otimes A(\mu)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{35}$$

$$\begin{pmatrix} D_{11}(\lambda) & D_{12}(\lambda) \\ D_{21}(\lambda) & D_{22}(\lambda) \end{pmatrix} \otimes (B_1(\mu)B_2(\mu))$$
$$= (B_1(\mu)B_2(\mu)) \otimes \begin{pmatrix} D_{11}(\lambda) & D_{12}(\lambda) \\ D_{21}(\lambda) & D_{22}(\lambda) \end{pmatrix} \frac{a(\lambda-\mu)}{b(\lambda-\mu)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{d}{a} & \frac{b}{a} & 0 \\ 0 & \frac{b}{a} & \frac{d}{a} & 0 \\ 0 & 0 & 0 & \frac{k}{a} \end{pmatrix}$$
$$- \frac{d(\lambda-\mu)}{b(\lambda-\mu)} (B_1(\lambda)B_2(\lambda)) \otimes \begin{pmatrix} D_{11}(\mu) & D_{12}(\mu) \\ D_{21}(\mu) & D_{22}(\mu) \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{36}$$

They are almost the same as that in the 'FBF' case; there is also a nested $SU(1|1)$ substructure whose $\check{r}$ matrix appears in equation (36). We obtain the following nested Bethe-ansatz equation,

$$\begin{aligned} \mathrm{e}^{\mathrm{i}k_j L} &= -\prod_{\alpha=1}^{M} \frac{k_j - \lambda_\alpha + \mathrm{i}c/2}{k_j - \lambda_\alpha - \mathrm{i}c/2} \\ 1 &= -\prod_{l=1}^{N} \frac{\lambda_\gamma - k_l - \mathrm{i}c/2}{\lambda_\gamma - k_l + \mathrm{i}c/2} \prod_{\alpha=1}^{M} \frac{\lambda_\gamma - \lambda_\alpha + \mathrm{i}c}{\lambda_\gamma - \lambda_\alpha - \mathrm{i}c} \prod_{\beta=1}^{M'} \frac{\mu_\beta - \lambda_\gamma + \mathrm{i}c/2}{\mu_\beta - \lambda_\gamma - \mathrm{i}c/2} \\ 1 &= -\prod_{\alpha=1}^{M} \frac{\mu_c - \lambda_\alpha - \mathrm{i}c/2}{\mu_c - \lambda_\alpha + \mathrm{i}c/2}, \end{aligned} \tag{37}$$

which were also derived in [10] by means of the coordinate Bethe-ansatz.

4. The ground state and its low-lying excitations

4.1. BFF case

In this case, the boson state is chosen as the reference state. There are N particles in all. After M lower operators B act on the reference state, there are just $N - M$ bosons. Analogously, after M' lower operators act on the second state, there are $M - M'$ fermions of species 1 and M' fermions of species 2. We follow the same analysis as in the other two cases. Taking the logarithm of (30), we have

$$\begin{aligned} k_j L &= 2\pi I_j + \sum_{l=1}^{N} \Theta_1(k_j - k_l) + \sum_{\alpha=1}^{M} \Theta_{-1/2}(k_j - \lambda_\alpha) \\ 2\pi J_\gamma &= \sum_{l=1}^{N} \Theta_{-1/2}(\lambda_\gamma - k_l) + \sum_{\beta=1}^{M'} \Theta_{1/2}(\lambda_\gamma - \mu_\beta) \\ 2\pi J'_c &= \sum_{\alpha=1}^{M} \Theta_{-1/2}(\mu_c - \lambda_\alpha) + \sum_{\beta=1}^{M'} \Theta_1(\mu_c - \mu_\beta) \end{aligned} \tag{38}$$

where $\Theta_n(x) = -2\tan^{-1}(x/nc)$ and I_j is an integer (half-odd integer) if $N - M - 1$ is even (odd), while J_γ is an integer (half-odd integer) if $N - M' - 1$ is even (odd), and J'_c is an integer (half-odd integer) if $M - M' - 1$ is even (odd). Once all roots $\{k_j, \lambda_\gamma, \mu_c\}$ are solved from

the above equations (38) for a given set of quantum numbers $\{I_j, J_\gamma, J'_c\}$, the energy and the momentum will be calculated by

$$E = \sum_{j=1}^{N} k_j^2, \qquad P = \frac{2\pi}{L}\left[\sum_{j=1}^{N} I_j - \sum_{\gamma=1}^{M} J_\gamma - \sum_{c=1}^{M'} J'_c\right]. \tag{39}$$

4.1.1. The ground state. It is easy to show that the right-hand side of the first equation in (38) is a monotonically increasing function of k_j, i.e., if $I_i < I_j$, then $k_i < k_j$. Thus the configuration of $\{I_j\}$ for the ground state is given by successive integers of half-integers symmetrically arranged around zero. Given a set of quantum numbers I_j, J_γ, J'_c with the solutions $k_j, \lambda_\gamma, \mu_c$, it is useful to consider the weak-coupling limit $c \to 0^+$. Due to $\Theta_{\pm n}(x) \to \mp\pi\ \mathrm{sgn}(x)$, equations (38) become

$$\begin{aligned} 2\pi I_j &= k_j L + \sum_{l=1}^{N} \pi\ \mathrm{sgn}(k_j - k_l) - \sum_{\alpha=1}^{M} \pi\ \mathrm{sgn}(k_j - \lambda_\alpha) \\ 2\pi J_\gamma &= \sum_{l=1}^{N} \pi\ \mathrm{sgn}(\lambda_\gamma - k_l) - \sum_{\beta=1}^{M'} \pi\ \mathrm{sgn}(\lambda_\gamma - \mu_\beta) \\ 2\pi J'_c &= \sum_{\alpha=1}^{M} \pi\ \mathrm{sgn}(\mu_c - \lambda_\alpha) - \sum_{\beta=1}^{M'} \pi\ \mathrm{sgn}(\mu_c - \mu_\beta). \end{aligned} \tag{40}$$

The subscripts of the rapidities $k_j, \lambda_\gamma, \mu_c$ are chosen in such a way that their quantum numbers $k_j, \lambda_\gamma, \mu_c$ are all ranged in increasing order. Then we have

$$\begin{aligned} 2(I_{j+1} - I_j - 1) &= (k_{j+1} - k_j)\frac{L}{\pi} - \sum_{\alpha=1}^{M}[\mathrm{sgn}(k_{j+1} - \lambda_\alpha) - \mathrm{sgn}(k_j - \lambda_\alpha)] \\ 2(J_{\gamma+1} - J_\gamma) &= \sum_{l=1}^{N}[\mathrm{sgn}(\lambda_{\gamma+1} - k_l) - \mathrm{sgn}(\lambda_\gamma - k_l)] - \sum_{\beta=1}^{M'}[\mathrm{sgn}(\lambda_{\gamma+1} - \mu_\beta) - \mathrm{sgn}(\lambda_\gamma - \mu_\beta)] \\ 2(J'_{c+1} - J'_c + 1) &= \sum_{\alpha=1}^{M}[\mathrm{sgn}(\mu_{c+1} - \lambda_\alpha) - \mathrm{sgn}(\mu_c - \lambda_\alpha)]. \end{aligned} \tag{41}$$

Thus, if $J'_{c+1} - J'_c = m$, there will be $m + 1$ solutions of λ_α between μ_c and μ_{c+1}, and if $I_{j+1} - I_j = n$, correspondingly with λ_α satisfying $k_j < \lambda_\alpha < k_{j+1}$, then we will get $k_{j+1} - k_j = \frac{2\pi n}{L}$; in contrast, with no λ_α between k_j and k_{j+1} we will have $k_{j+1} - k_j = \frac{2\pi(n-1)}{L}$. Obviously, such a λ_α always repels the k rapidity away, leading to a rise in the energy. Thus the ground state of this system should have no λ_α lying in k_j.

In the strong-coupling limit $c \to \infty$, we have $\tan^{-1}\frac{x}{c} \simeq \frac{x}{c}$. Substituting these into the secular equations (38) for the ground state ($M = 0$, $M' = 0$) and the low-lying excited state ($M = 1$, $M' = 0$), the secular equations become

$$k_j L = 2\pi I_j - 2\sum_{l=1}^{N} \frac{k_j - k_l}{c} \qquad k'_j L = 2\pi I'_j - 2\sum_{l=1}^{N} \frac{k'_j - k_l}{c} + 2\frac{k'_j - \lambda_1}{c/2}.$$

Here we change the ground state by adding one λ_α which leads to $I_j - I'_j = 1/2$. From the two equations above we can get

$$(k_{j+1} - k_j)L\left[1 + \frac{2N}{cL}\right] = 2\pi \qquad (k'_{j+1} - k'_j)L\left[1 + \frac{2N-4}{cL}\right] = 2\pi. \tag{42}$$

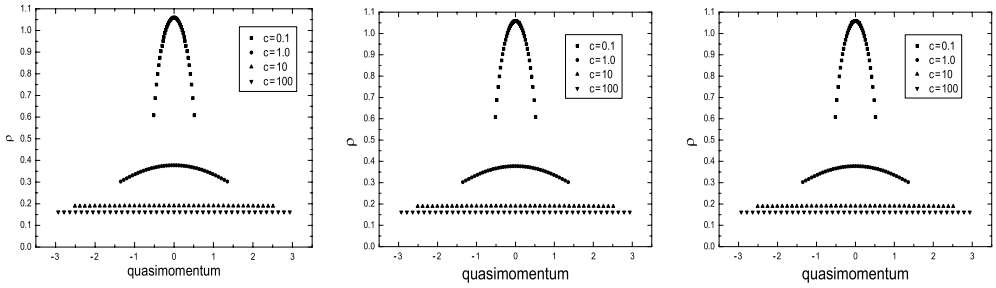

Figure 1. The density of state in k space for the ground state (left), adding one fermion (middle), and adding three fermions (right). The distribution changes from a histogram to a narrow peak gradually for the coupling from strong to weak. The figure is plotted for $c = 100, 10, 1, 0.1$.

In order to analyse the low-lying excited characters of the system more conveniently, we introduce density of roots

$$\rho(k_j) = \frac{1}{L(k_{j+1} - k_j)}, \qquad \sigma(\lambda_\gamma) = \frac{1}{L(\lambda_{\gamma+1} - \lambda_\gamma)}, \qquad \omega(\mu_c) = \frac{1}{L(\mu_{c+1} - \mu_c)}.$$

In the thermodynamics limit, we have $\rho(k) = \frac{1}{L}\frac{\mathrm{d}I(k)}{\mathrm{d}k}$, corresponding to $\sigma(\lambda) = \frac{1}{L}\frac{\mathrm{d}J(\lambda)}{\mathrm{d}\lambda}$ and $\omega(\mu) = \frac{1}{L}\frac{\mathrm{d}J'(\mu)}{\mathrm{d}\mu}$. In terms of these densities, the energy and the momentum per length are given by

$$E/L = \int k^2\rho(k)\,\mathrm{d}k, \qquad P/L = \int k\rho(k)\,\mathrm{d}k \tag{43}$$

while N, M and M' are determined by

$$N/L = \int \rho(k)\,\mathrm{d}k, \qquad M/L = \int \sigma(\lambda)\,\mathrm{d}\lambda, \qquad M'/L = \int \omega(\mu)\,\mathrm{d}\mu \tag{44}$$

where $K_n(x) = \frac{1}{\pi}\frac{nc/2}{n^2c^2/4+x^2}$ the density of the state satisfies the integral equation

$$\rho_0(k) = \frac{1}{2\pi} + \int_{-k_F}^{k_F} K_2(k - k')\rho_0(k')\,\mathrm{d}k' \tag{45}$$

in the thermodynamic limit, where $\rho_0(k)$ and k_F are the density and integration limit for the ground state, respectively. We solved the secular equation for 42 particles with $M = M' = 0$ numerically, and the density of the ground state is depicted in figure 1 (left) for different coupling constants.

Comparing with the ground state, we plot the spectrum for the low-lying excitation ($M = 1$, $M' = 0$) in figure 1 (middle). The density of state is slightly compressed compared with figure 1 (middle). It is not obvious from numerical results in figure 1 (middle) compared to figure 1 (left), so we increase M from 1 to 3, and the curves are depressed more evidently in the system with $N = 42$ (figure 1 (right)). As the value of M rises, the number of fermions rises correspondingly, and the larger the number of fermions, the higher the energy should be. As a result, the ground state contains only bosons which agrees with the results of our asymptotic analysis.

4.1.2. Particle–hole excitation. The quantum numbers for the ground state in the N-particle system are $\{I_j\} = \{-(N-1)/2, \ldots, (N-1)/2\}$, $\{J_\gamma\} = \{J'_c\} = \textit{empty}$. If we add a hole to the ground state, then the quantum numbers take the values $I_1 = -(N-1)/2 + \delta_{1,j}$ for

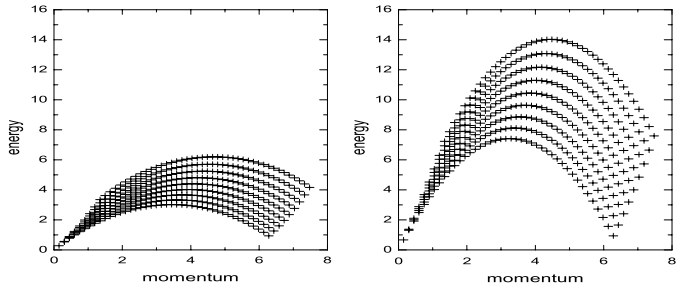

Figure 2. Particle–hole excitation for $c = 1.0$ (left) and $c = 10.0$ (right).

$(1 \leqslant j_1 \leqslant N)$, $I_j = I_{j-1} + \delta_{j,j_1}$ for $(j = 2, \ldots, N-1)$, $I_N = I_0$, $(I_0 > (N-1)/2)$; we call it the particle–hole excitation. In figure 2 the excitation spectrum is plotted with coupling numbers $(c = 1.0, 10.0)$.

In the thermodynamic limit, we use the expression $\rho(k) = \rho_0(k) + \rho_1(k)/L$, then removing one I from the original symmetric sequence and adding a new I_n outside it, we have

$$\rho_1(k) + \delta(k - \overline{k}) = \int K_2(k - \overline{k})\rho_1(k')\,\mathrm{d}k' + K_2(k - k_p). \tag{46}$$

The excited energy consists of two terms $\Delta E = \frac{1}{L}\int \rho_1(k)k^2\,\mathrm{d}k + \frac{1}{L}k_p^2 = \xi_h(\overline{k}) + \xi_a(k_p)$, where ξ_h is holon's energy and $\xi_a(k_p)$ is particle's energy, and they can be calculated by

$$\begin{aligned} \rho_1^h(k, \overline{k}) &= \int K_2(k - k')\rho_1^h(k' - \overline{k})\,\mathrm{d}k' - K_2(k - \overline{k}) \\ \xi_h(\overline{k}) &= -\overline{k}^2 + \int_{-k_F}^{k_F} k^2\rho_1^h(k, \overline{k})\,\mathrm{d}k. \end{aligned} \tag{47}$$

4.1.3. Adding one fermion. If we add one fermion into the ground state, this excitation can be characterized by moving the quantum number J_1 in the following region:

$$-(N-1)/2 < J_1 < (N-1)/2.$$

We describe this phenomenon in figure 3(*a*). Replacing one boson by one fermion corresponds to a 2-parameter excitation. Its energy is given by $\Delta E = \int k^2\rho_1^c(k, \lambda)\,\mathrm{d}k$ with $\rho_1(k)$ solving

$$\rho_1(k) + \delta(k - \overline{k}) = \int K_2(k - k')\rho_1(k')\,\mathrm{d}k' + K_1(k - \lambda_1). \tag{48}$$

Then we have one-fermion excitation $\Delta E = \xi_h(\overline{k}) + \xi_c(\lambda)$, where $\xi_h(\overline{k})$ is the same as equations (47) and ξ_c is defined by $\xi_c(\lambda) = \int k^2\rho_1^c(k, \lambda)\,\mathrm{d}k$ with

$$\rho_1^c(k, \lambda) = \int_{-k_F}^{k_F} K_2(k - k_l)\rho_1^c(k_l, \lambda)\,\mathrm{d}k_l - K_1(k - \lambda_1). \tag{49}$$

4.1.4. Two-fermion excitation. If two spin-up fermions or two spin-down fermions are permitted in this system, then the arrangement of their quantum numbers is

$$-(N-1)/2 \leqslant J_1 \leqslant J_2 \leqslant (N-1)/2.$$

Numerical calculation for this type of excitation is shown in figure 3(*b*). Furthermore, if we introduce one spin-up and one spin-down fermions in this system, the results are the same as figure in 3(*b*). Comparing to the system of pure bosons, we found that such excitation is similar to isospinon–isospinon excitation in a two-band $SU(2)$ system [24].

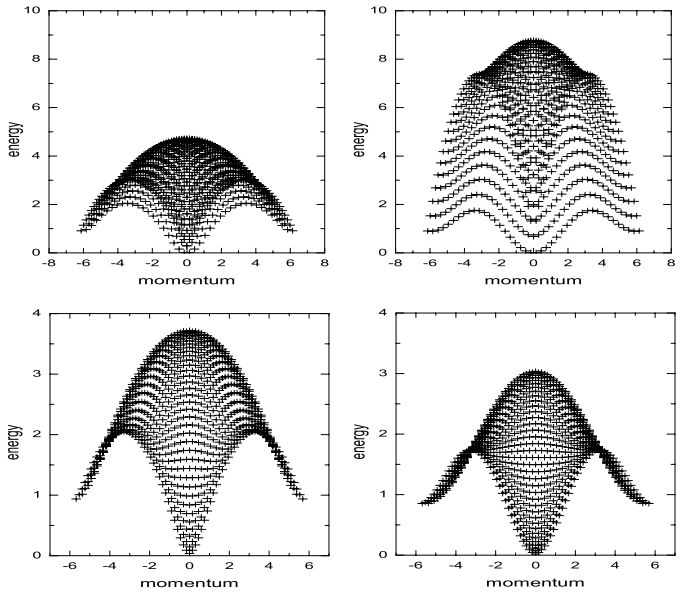

Figure 3. Top panels show the one-fermion excitation for $c = 1.0$ (left) and $c = 10.0$ (right); bottom panels show the two-fermion excitation for $c = 1.0$ (left) and $c = 10.0$ (right).

4.2. FBF case

From equations (34), we know that there are $N - M$ fermions of species 1, $M - M'$ bosons and M' fermions of species 2. Taking the logarithm of equations (34), we get

$$\begin{aligned} k_j L &= 2\pi I_j + \sum_{\alpha=1}^{M} \Theta_{1/2}(k_j - \lambda_\alpha) \\ 2\pi J_\gamma &= \sum_{l=1}^{N} \Theta_{-1/2}(\lambda_\gamma - k_l) + \sum_{\beta=1}^{M'} \Theta_{1/2}(\lambda_\gamma - \mu_\beta) \\ 2\pi J'_c &= \sum_{\alpha=1}^{M} \Theta_{-1/2}(\mu_c - \lambda_\alpha). \end{aligned} \tag{50}$$

The quantum number I_j takes integer or half-integer values, depending on whether M is even or odd, and J'_c take half-integer (integer) values when M is even (odd). Whereas J_γ is integer (half-integer), if $N - M' - 1$ is even (odd). In the weak-coupling limit, $c \to 0^+$, $\Theta_{1/2}(x) \to -\pi\,\mathrm{sgn}(x)$, and $\Theta_{-1/2}(x) \to \pi\,\mathrm{sgn}(x)$ for $x \gg 1$, hence equations (50) become

$$\begin{aligned} 2I_j &= \frac{k_j L}{\pi} + \sum_{\alpha=1}^{M} \mathrm{sgn}(k_j - \lambda_\alpha) \\ 2J_\gamma &= \sum_{l=1}^{N} \mathrm{sgn}(\lambda_\gamma - k_l) - \sum_{\beta=1}^{M'} \mathrm{sgn}(\lambda_\gamma - \mu_\beta) \\ 2J'_c &= \sum_{\alpha=1}^{M} \pi\,\mathrm{sgn}(\mu_c - \lambda_\alpha) - \sum_{\beta=1}^{M'} \mathrm{sgn}(\mu_c - \mu_\beta). \end{aligned} \tag{51}$$

On choosing I_j, J_γ, J'_c in an increasing order, for a given M and $M \leqslant N$ with the rules of Young tableau, the minimum value on the left-hand side of the third equation of equations (51) is $-M+2$. Therefore, the smallest λ_α must be smaller than the smallest μ_c. Otherwise the left-hand side would be $-N$, and if we take the maximum value of the left-hand side $M-2$, correspondingly the largest λ_α must be larger than the largest μ_c. In other words, the presence of μ_c is only allowed in λ space. Furthermore, we can obtain $|J_c| \leqslant (N-M')/2$, when there is no spin-down fermion ($M'=0$); all J_γ are likely to stay in the I_j sequence.

Supposing the rapidities k_j, λ_γ, μ_β are in increasing order as I_j, J_γ, J'_c, then we have

$$\begin{aligned}
2(I_{j+1}-I_j) &= \frac{(k_{j+1}-k_j)L}{\pi} - \sum_{\alpha=1}^{M}[\operatorname{sgn}(k_{j+1}-\lambda_\alpha)-\operatorname{sgn}(k_j-\lambda_\alpha)] \\
2(J_{\gamma+1}-J_\gamma) &= \sum_{l=1}^{N}[\operatorname{sgn}(\lambda_{\gamma+1}-k_l)-\operatorname{sgn}(\lambda_\gamma-k_l)] - \sum_{\beta=1}^{M'}[\operatorname{sgn}(\lambda_{\gamma+1}-\mu_\beta)-\operatorname{sgn}(\lambda_\gamma-\mu_\beta)] \\
2(J'_{c+1}-J'_c) &= \sum_{\alpha=1}^{M}[\operatorname{sgn}(\mu_{c+1}-\lambda_\alpha)-\operatorname{sgn}(\mu_c-\lambda_\alpha)].
\end{aligned} \tag{52}$$

Therefore, let $I_{j+1}-I_j=n$ and there will be $k_{j+1}-k_j=2\pi(n-1)/L$ existing λ_α between k_j and k_{j+1}. Otherwise $k_{j+1}-k_j=2\pi n/L$. This means that existing λ_α in k_j space will decrease the system's energy, so the ground state should have more λ_α. If $J'_{c+1}-J'_c=m$, we know there will be m solutions λ_α satisfying $\mu_c<\lambda_\alpha<\mu_{c+1}$. When $J_{\gamma+1}-J_\gamma=m'$, m' is the integer which equals or surpasses 1, then there must be some k_j lying in neighbouring λ_γ, and the larger the number of μ_β between λ_γ and $\lambda_{\gamma+1}$ we have, the larger the number of k_j we hope. If m' is large enough, we wish that k_j be large enough too, and μ_β be small enough. When m' equals 1, there is only one k_j between λ_γ and $\lambda_{\gamma+1}$, this state contains no μ_β; that is to say k and λ alternate.

Then in the strong-coupling limit $c\to\infty$, $\tan^{-1}(x/c)\to x/c$ and equations (50) give rise to

$$k_jL = 2\pi I_j - 2M\frac{k_j-\lambda_\alpha}{c/2}.$$

Furthermore, the new form will be

$$(k_{j+1}-k_j)L\left[1+\frac{4M}{cL}\right]=2\pi. \tag{53}$$

From the formula above, we know if M approaches N, $\Delta k=k_{j+1}-k_j$ is the smallest, so is the energy. Therefore, the $M=N$ state is the ground state.

We describe the density of ground state by numerical approaches, and find that this is the same as figure 1 (left). If we permit $M=N-3$, i.e., there are three spin-up fermions lying in the ground state, the density of state is also shown in figure 1 (right). It is obvious that the more the fermions lying in this system, the higher the energy will be. Thus the ground state is exactly all bosons without any fermion, which verifies the analysis in case 1. Furthermore, we remove one of the I' from the ground state sequence and add a 'new' I_0 outside. Excited states are obtained by varying the quantum number as

$$I_j = -(N-1)/2,\ldots,-(N-1)/2+i-1,-(N-1)/2+i+1,\ldots,(N-1)/2,I_0,$$
$$J_\gamma = -(N-1)/2,\ldots,(N-1)/2,$$

and consequently we get the excitation spectrum which is not different from figure 2.

For this case, the excitation of adding one fermion is obtained from $M = N - 1$, which contains one free parameter in the I sequence and one free parameter in the J sequence. Thus the order of quantum numbers is

$$I_1 = -N/2 + 1 + \delta_{j_1,1} \qquad I_j = I_{j-1} + 1 + \delta_{j_1,j} \qquad (j = 2, \ldots, N)$$
$$J_1 = -N/2 + \delta_{\alpha_1,1}, \qquad J_\alpha = J_{\alpha-1} + 1 + \delta_{\alpha,\alpha_1} \qquad (\alpha = 2, \ldots, M)$$

where $1 \leqslant j_1 \leqslant N + 1$, $1 \leqslant \alpha_1 \leqslant M + 1$. The excitation spectrum is shown in figure 3(*a*), which is also consistent with the excitation of adding one fermion in case 1.

Comparing to case 1, when we add two spin-up fermions ($M = N - 2$), there are two free parameters in the J sequence. The result is depicted in figure 3(*b*).

4.3. FFB case

There are $N - M$ fermions of species 1, $M - M'$ fermions of species 2 and M' bosons in equations (37). Also taking the logarithm of these equations we obtain

$$\begin{aligned} k_j L &= 2\pi I_j + \sum_{\alpha=1}^{M} \Theta_{1/2}(k_j - \lambda_\alpha) \\ 2\pi J_\gamma &= \sum_{l=1}^{N} \Theta_{-1/2}(\lambda_\gamma - k_l) + \sum_{\alpha=1}^{M} \Theta_1(\lambda_\gamma - \lambda_\alpha) + \sum_{\beta=1}^{M'} \Theta_{-1/2}(\lambda_\gamma - \mu_\beta) \\ 2\pi J'_c &= \sum_{\alpha=1}^{M} \Theta_{-1/2}(\mu_c - \lambda_\alpha) \end{aligned} \tag{54}$$

where I_j is an integer (half-odd integer) if $N - M - 1$ is even (odd), J_γ is an integer (half-odd integer) if $N - M - M' - 1$ is even (odd), and J'_c is an integer (half-odd integer) if $M - 1$ is even (odd).

Considering the weak-coupling limit $c \to 0^+$, we have

$$\begin{aligned} 2(I_{j+1} - I_j) &= \frac{(k_{j+1} - k_j)L}{\pi} + \sum_{\alpha=1}^{M} [\mathrm{sgn}(k_{j+1} - \lambda_\alpha) - \mathrm{sgn}(k_j - \lambda_\alpha)] \\ 2(J_{\gamma+1} - J_\gamma + 1) &= \sum_{l=1}^{N} [\mathrm{sgn}(\lambda_{\gamma+1} - k_l) - \mathrm{sgn}(\lambda_\gamma - k_l)] \\ &\quad + \sum_{\beta=1}^{M'} [\mathrm{sgn}(\lambda_{\gamma+1} - \mu_\beta) - \mathrm{sgn}(\lambda_\gamma - \mu_\beta)] \\ 2(J'_{c+1} - J'_c) &= \sum_{\alpha=1}^{M} [\mathrm{sgn}(\mu_{c+1} - \lambda_\alpha) - \mathrm{sgn}(\mu_c - \lambda_\alpha)]. \end{aligned} \tag{55}$$

If we set $J'_{c+1} - J'_c = m$, there will be m $\lambda'_\alpha s$ satisfying $\mu_c < \lambda_\alpha < \mu_{c+1}$. Letting $I_{j+1} - I_j = n$, if there is a λ_α satisfying $k_j < \lambda_\alpha < k_{j+1}$, we will get $k_{j+1} - k_j = 2\pi(n-1)/L$, otherwise $k_{j+1} - k_j = 2\pi n/L$. From those above, we find that adding a λ_α into the k_j space will expand the distance of neighbouring particles and lead to decrease of the energy. Thus the more λ_α lying between k_j and k_{j+1}, the lower the energy will be. Letting $J_{\gamma+1} - J_\gamma = n'$, there will be $n' + 1$ solutions of k_l and μ_β satisfying $\lambda_\gamma < k_l$, $\mu_\beta < \lambda_{\gamma+1}$, where n' is the integer which equals or surpasses 1, i.e., $n' = 1$. Then there will be two fermions or two bosons or one fermion and one boson between λ_γ and $\lambda_{\gamma+1}$. As we always set the distance between the neighbouring quantum numbers as 1, the most reasonable state should be one fermion and one boson between λ_γ and $\lambda_{\gamma+1}$.

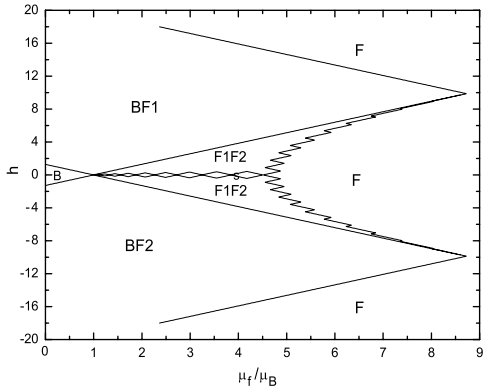

Figure 4. The phase diagram of free particles. The vertical axis stands for the external magnetic field, and the horizontal axis stands for the ratio of chemical potential between bosons (μ_b) and fermions (μ_f). The region 'B' stands for the state with merely bosons. 'BF1' means the mix state of bosons and the spin-up fermions. 'BF2' means the mix state of bosons and the spin-down fermions. 'F1F2' stands for the mix state of bosons and two kinds of fermions, but $N_\uparrow \neq N_\downarrow$. The region 'S' means the spin singlet state, i.e., $N_\uparrow = N_\downarrow$. The region 'F' means that there are no bosons in the ground state.

Therefore, the state of $M = N, M' = N$ is a boson state which has the lowest energy. This result coincides with parts of Lai's [10] results. The density of the ground state is the same as figure 1 (left). When $M' = N - 1, M = N$, i.e., there is one spin-up fermion in the ground state, we have the same result as figure 1 (middle). And the more spin-up fermions there are, the higher the energy should be. The particle–hole excitation spectrum is as in figure 2. Due to the restriction given by the Young tableau, $M = N - 2$ gives the excitation of adding one spin-up and one spin-down fermion, and exactly there are three holes in the J sequence. This excitation is depicted in figure 3(b). It also sustains the analysis of case 1.

4.4. The consistency of the three cases

In our analytic and numerical results, the density of ground state (figure 1 (left)) and the energy–momentum spectrum of low-lying excitations (figure 3(a) and (b)) in the three cases are the same, and the ground states for all the three cases are the states with merely bosons; i.e., in the BFF case, the ground state is '$M = M' = 0$', '$M = N, M' = 0$' for the FBF case and '$M = M' = N$' for the FFB case, they are all merely boson states. Thus we conclude that the properties of this multi-component system in one dimension are independent of the reference state which we choose, which manifests the consistency of these cases. This kind of equivalence was noticed in the so-called supersymmetry t–J model [25].

5. The ground state phase diagram in the presence of external magnetic fields

We take into account both the chemical potentials and the external magnetic fields which bring about the Zeeman splitting for fermions. First, we analyse two limit cases: the case of free particles and the case of particles with infinitely strong interaction. We have obtained the phase diagram as figures 4 and 5.

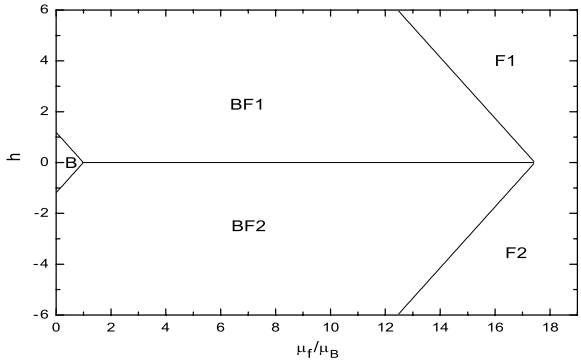

Figure 5. The phase diagram of particles with the strong-coupling limit $c \to \infty$.

5.1. Weak-coupling limit

When $c \to 0$, we can treat these particles as free particles. Thus in the ground state, only fermions have non-zero momentum because of the Pauli excluding principle. We plot the phase diagram in figure 4 for 42 particles. The region 'S' comprises some rhombi that stands for different numbers of Fermi pairs $(1, 3, 5, 7, \ldots)$. We take two rhombi (3 and 5 pairs of fermions) for example. In the valley region between them (here we consider $h > 0$), there are 5 spin-up and 3 spin-down fermions. Thus on the left boundary of this valley, the energy of 3 pairs of spin singlet fermions equals that of 5 spin-up plus 3 spin-down fermions. Because there are $N - M$ bosons, $M - M'$ spin-up fermions, and M' spin-down fermions,

$$\begin{aligned} H_{3\uparrow 3\downarrow} &= E_{33} - \mu_B \left((N - M) + \frac{\mu_f}{\mu_B} M \right) - \frac{h}{2}(M - 2M') \\ &= 2\left(\frac{L}{2\pi} \right)^2 - \mu_B \left(36 + 6\frac{\mu_f}{\mu_B} \right). \end{aligned} \tag{56}$$

Analogously,

$$\begin{aligned} H_{5\uparrow 3\downarrow} &= E_{53} - \mu_B \left((N - M) + \frac{\mu_f}{\mu_B} M \right) - \frac{h}{2}(M - 2M') \\ &= 12\left(\frac{L}{2\pi} \right)^2 - \mu_B \left(34 + 8\frac{\mu_f}{\mu_B} \right) - h. \end{aligned} \tag{57}$$

Thus the phase boundary can be described as $H_{3\uparrow 3\downarrow} = H_{5\uparrow 3\downarrow}$, which reduces to $2\triangle + h = 10\left(\frac{L}{2\pi}\right)^2$. Similarly, the equation of the right boundary is $2\triangle - h = 8\left(\frac{L}{2\pi}\right)^2$ in which $\triangle = \mu_f - \mu_B$.

5.2. Strong-coupling limit

When the intensity of interaction approaches infinity, any two particles cannot stay in the same state. Thus there should be a much stronger excluding force than the Pauli principle. Because the energy of N particles ($N - M$ bosons, $M - M'$ spin-up fermions, M' spin-down fermions) is $H = E_0 - \mu_B\left((N - M) + \frac{\mu_f}{\mu_B} M\right) - \frac{h}{2}(M - 2M')$, when the direction of the magnetic field is along the spin-up ($h > 0$) direction, the spin-down fermions cannot appear in the ground state. Therefore, in region 'BF1', there are only fermions with spin up and bosons; and in region 'F1', there are N fermions with spin up. The same analysis holds when $h < 0$.

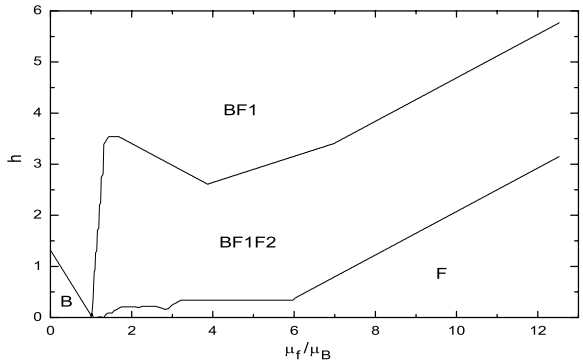

Figure 6. A sketch of phase when interaction strength $c = 1$.

5.3. General case

In the case of general coupling, this model satisfies equation (30). We solve it within the rules of the Young tableau ($N - M \geqslant M - M' \geqslant M''$), thus the case of all the particles being fermions in the ground state cannot be achieved here. And we only computed the case of $h > 0$ because of symmetry. As shown in figure 6, the phase boundary between 'B' and 'BF1' is the same as that in the two limit cases above, which intersects with the h-axis at $h = 2\mu_B$. The boundary between 'BF1' and 'BF1F2' can be divided into two parts from $\mu_f/\mu_B \approx 1.5$. The left part has some fluctuations which means adding a pair of spin-up fermions on 'BF1F2' side. The highest point ($\mu_f/\mu_B \approx 1.5$) stands for 19 spin-up and 1 spin-down fermion. According the Young tableau, it cannot add a spin-up fermion pair on the '19 up 1 down' state any more. Thus when $\mu_f/\mu_b > 1.5$, we can only add the spin-down Fermi pair. This is the reason why there is an inflexion at $\mu_f/\mu_B \approx 1.5$. We figure out an outline of the spin singlet phase 'F', in which we can also see that there are some small fluctuations which stand for adding a pair of spin singlet fermions. It is somewhat similar to the free-particle case. Because of the Pauli excluding principle, only two fermions with different spins can stay in the same state. Hence in the spin singlet phase, the pair numbers are $1, 3, 5, \ldots$ along the horizontal axis. From $\mu_f/\mu_B \approx 6$ on, the pair number equals 14, which is the largest pair number we can get here.

6. Summary and discussion

In summery, we have explicitly derived the Bethe-ansatz equation for the model of one-dimensional Bose–Fermi mixture by means of QISM. We analysed the properties of the ground state and the low-lying excitations on the basis of the Bethe-ansatz equations. We found that the ground state of this system is the state with merely bosons. The low-lying excitations were discussed extensively. The energy–momentum spectrum for three types of excitations, holon–particle, one fermion, two fermions, were plotted for $c = 10.0$ and $c = 1.0$. We discussed the phase diagram of the ground state in the presence of external magnetic fields and chemical potentials, from which we can know about the populations of bosons and fermions at a given magnetic field and chemical potential.

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Appendix. Permutation matrices and the generators of $SU(3)$ Lie algebra

For the BFF case, the permutation matrix reads

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{A.1}$$

For the FBF case, it reads

$$P_2 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{A.2}$$

For the FFB case, it reads

$$P_3 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{A.3}$$

The generators for $SU(3)$ Lie algebra are given by

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$E_{\alpha_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_{\alpha 2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_{\alpha_1+\alpha_2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_{-\alpha_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_{-\alpha 2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad E_{-(\alpha_1+\alpha_2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

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